Cold Strongly Coupled Atoms Make a Near-perfect Liquid

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A system of cold trapped atoms near the Feshbach resonance with a large scattering length has attracted much attention lately. We suggest that the transport properties of the atomic cloud reveal its structure better than its equation of state. Universality as applied to cold Fermi systems with density n means that their viscosity is $\eta = \hbar n \alpha_\eta$, where α_η is a constant. Using the Heisenberg uncertainty principle and Einstein's relation between diffusion and viscosity we derive a lower bound for this constant: $\alpha_\eta \geq (6\pi)^{-1}$. We rederive the bound established recently in the context of strongly coupled hot supersymmetric gauge theories. We extract the empirical value of the viscosity from data on damping of small vibrations of a trapped atomic cloud. The minimal value is $\alpha_\eta|_{min} \approx 0.5$. We show that this result is much lower than any Fermi-gas-based picture can produce with standard Pauli-blocked binary collisions. We conclude that near the Feshbach resonance the system is liquid-like.

I. INTRODUCTION

Recently there has been a growing interest in several different areas of physics in strongly interacting systems, with dimensionless interaction parameter large or even formally infinite. In this paper we will discuss mostly one example: **(i)** trapped cold atomic Fermi gases with large scattering length in the vicinity of the Feshbach resonance; another is **(ii)** a strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) gauge theory, a four-dimensional conformal field theory (CFT). There are two more known examples of this kind which we will *not* discuss here: **(iii)** the strongly coupled quark-gluon plasma [1] (sQGP) which was found in heavy ion experiments at RHIC at temperatures above the critical $T = (1-2)T_c = 170 - 350\, MeV$; **(iv)** the usual QED plasma. Its classical one-component version is particularly well studied, and for strong coupling, *i.e.* $\Gamma_{\rm plasma} = (Ze)^2 n^{1/3}/T = 2 - 300$, is known to be a liquid with a viscosity reaching its minimum value at $\Gamma_{\rm plasma} \sim 10$ (see e.g. [2]).

What unites all of these systems is their transport properties, which are dramatically different from weak coupling extrapolations. The difference between weak and strong coupling is also present in the total energy (or free energy) but it is much less dramatic.

In fact, all of these four systems behave as very good liquids, with an (analog of) the mean free path being even smaller than the inter-particle distance. This prompts us to ask:

Thinking why can it be possible leads to general questions we will address in this paper:

(1) What is *the most perfect* liquid?

(2) Is a system of cold strongly coupled Fermions a perfect liquid?

Let us now introduce these two systems in more detail beginning with the case **(ii)**. The CFT is a theoretical toy model, mostly used in the context of string theories. It is a model resembling QCD, the gauge theory of strong interaction, in which the coupling is not "running" with scale. Thus, the strong coupling regime is reached simply by putting the coupling constant in the Lagrangian $\lambda_s = g^2 N$ to be large where g is the gauge coupling and N is the number of colors. Only this combination appears if N is large. This limit can be addressed using the AdS/CFT correspondence as originally suggested by Maldacena [3], whereby the quantum intricacies of the strongly coupled gauge theory are mapped on a classical problem in gravity albeit in ten dimensions. The finite temperature version of this theory describes a plasma-like phase with strongly coupled constituents. The four dimensional world (in which the CFT fields live) is a surface in ten dimensional space, at some distance from a black hole, with a mass adjusted to yield the desired temperature T at this surface. From this conclusion stem the following two results One is the equation of state of the underlying gauge theory at strong coupling $\lambda_s \gg 1$ [4]

$$\frac{p_{\lambda_s}(T)}{p_0(T)} = 1 - \frac{1}{4} + \mathcal{O}(\lambda_s^{-3/2})\,, \tag{1}$$

where $p_0(T) \sim T^4$ is the Stephan-Boltzmann pressure for zero coupling. The second is the viscosity of the underlying gauge theory at strong coupling [5]

$$\lim_{\lambda_s \to \infty} \frac{\eta}{\hbar\, s} = \frac{1}{4\pi}\left(1 + \mathcal{O}(\lambda_s^{-3/2})\right)\,, \tag{2}$$

given in units of the free entropy density s^*. The corrections were recently calculated in [6].

Note that while the pressure is only changed by 1/4 when one changes coupling all the way from zero to infinity, the viscosity $\eta/(\hbar s)$ changes from infinity to a finite (and surprisingly small) number. One thus may wonder whether (i) other strongly coupled systems show similar behavior, and (ii) whether such limiting numbers can be universal and theoretically understood without explicit calculations.

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*Thermal gauge theories are like blackbody radiation, there is no ordinary density but only entropy density $s \sim T^3$.

The holographic principle in the Maldacena limit and the Kubo formula show that the viscosity is proportional to the graviton absorption cross section in bulk by the black hole, while (according to Beckeinstein-Hawkins argument) the free entropy is related to its area. As a result the same limit for the viscosity holds for a number of backgrounds, even in different space dimensions. These observations prompted [7] to conjecture (2) as a universal lower bound valid for any thermal system in strong coupling. Below we provide its heuristic derivation using the uncertainty relation and Einstein's famous relation between the diffusion constant and the fluid viscosity. As a result, we show how the bound (2) fits well into the the liquid-like picture of CFT at finite temperature. We then use these insights to derive an even lower bound for cold Fermi systems, and conjecture that at strong coupling they also make universal near-perfect liquids.

Trapped Bose and Fermi atoms at low temperature in a regime of a large scattering length, $a \to \pm\infty$, have recently attracted significant experimental and theoretical attention. This limit can be reached via the Feshbach resonance at zero energy. The universality of this limit for the bulk parameters of the trapped atoms has been emphasized by a number of authors, see e.g. [8]. Basically it is a simple dimensional argument. As the only interaction parameter a gets infinite and thus drops out from all answers, there can be only one relevant scale of length (time, etc.). Thus all properties of matter follow by dimensional considerations modulo unknown constants. For example, the zero temperature pressure (energy density) of trapped atoms of density n can only be proportional to that of the ideal Fermi gas

$$\frac{p_\infty(n)}{p_0(n)} = (1-\beta) \tag{3}$$

where $p_0 \sim n^{5/3}/m$ is the ideal gas pressure for $a = 0$, p_∞ is the pressure at the Feshbach resonance, and β is some unknown universal dimensionless constant pertaining to the ultimate theory. For cold atoms, $\beta \approx 0.5$, as experiments and theoretical models show. We will not discuss it in this work. All we want to emphasize here is that the modification is not that dramatic.

The point we want to make by this paper is that the calculations and/or measurements of *transport properties* such as the diffusion coefficient and viscosity should be more revealing than those of the equation of state, as for them the strong coupling results should be radically different from the results in weak coupling, set by the mean free path of (quasi)particles.

The paper is structured as follows. In section II we discuss what universality would mean for the viscosity of a system of cold trapped atoms. Section III is devoted to the question of how small the viscosity can possibly be, based on general principles like the uncertainty relation. We then use the experimental data on damping of the low frequency vibrational mode in section IV to deduce the empirical viscosity value, to be compared with naive kinetic theory in section V.

II. LIQUID-LIKE PROPERTIES AND UNIVERSALITY

In the standard (weak coupling) regime of small scattering length a, transport properties are related with the particle mean-free path

$$l_{\rm mfp} = \frac{1}{n\sigma}\,, \qquad \sigma = 4\pi a^2 \,. \tag{4}$$

In the strong coupling regime $a \to \infty$ (4) gets meaningless, but since the cross section does not diverge but is bounded by unitarity, this regime was sometimes called "unitarity limited" one. Indeed, the maximal possible pair cross section is limited by $\sigma_{\rm max} = 4\pi/k^2$ for fixed collision energy (k is the wavenumber of relative motion). Thus, one may think that the mean free path is actually $l_{min} = 1/n\sigma_{\rm max}$. However, this is too naive since at strong coupling there is no reason to limit kinetics to a picture of propagating particles rarely suffering only binary collisions.

Instead, one should think in terms of the picture of a densely packed liquid. Cold gasses should have particles effectively localized in space at a scale $1/n^{1/3}$, the inter-particle distance, as it is the only length scale available. A classical strongly coupled system at zero temperature must crystallize, but quantum mechanics makes it possible to keep matter in the form of a liquid. Its kinetic description based on binary particle collisions via the Boltzmann equation does not make sense at strong coupling.

The adequate tool to describe low frequency dynamics of such matter is instead *viscous hydrodynamics*. As it is based on the expansion in *inverse* powers of the cross sections, or the expansion in small mean free paths l_{mfp}, the stronger the interaction the better this approach. The viscosity is in general defined via the dissipative part of the stress tensor and can be defined without any assumption on the underlying matter. It appears in observables like the sound dispersion law

$$\omega = c_s k + \frac{i}{2}\frac{4\eta}{3mn}k^2 \tag{5}$$

and thus can be measured. In fact below we will derive and use the analog of such expression for trapped atoms (rather than homogeneous infinite matter).

For cold fermionic systems there should be universal relations for transport properties, in a form similar to (1) and (3). In particular, zero temperature strongly coupled systems of nonrelativistic particles should have a viscosity proportional to their density n

$$\eta = n\hbar\alpha_\eta \,, \tag{6}$$

where α_η is some universal dimensionless coefficient. Similarly, the only time scale is given by the Fermi frequency $\tau^{-1} \sim \epsilon_F/\hbar \sim \hbar n^{2/3}/m$, where ϵ_F is the Fermi energy. Thus, the analogs of scattering rates should be proportional to this scale and so on.

The next question is what are the magnitude of these dimensionless parameters. If they are all of the order one, it would imply that the usual picture of Fermi gas, with particles localized in momentum-energy space and completely delocalized in space breaks down. Thus, particles are not well localized neither in momentum nor in coordinate space.

The dimensionless parameters can also be quite different from order one, such as in (2) which means that CFT in strong coupling is not just a liquid but an exceptionally perfect liquid. For atomic systems one may also think that as the interaction strength is driven to infinity the transport parameters such as the viscosity gets as small as possible. Before we address the specific bounds and data on the transport parameters of the strongly coupled liquid, it is worth emphasizing the empirical evidence for the near-perfect liquid.

A spectacular manifestation of the hydrodynamical behavior is the 'elliptic flow' seen in the trapped atomic systems after the trap is switched off [9]. The flow is seen also at high enough temperatures, so it is not by itself related to condensation or superfluidity. Good quantitative description of the data were obtained in a framework of ideal hydrodynamics, *without* any viscous terms. In principle,this comparison by itself may provide some upper limits on the viscosity. However, more quantitative evidences for the hydrodynamical behavior are given by the observed oscillations with a very small damping width which we will discuss in detail in section IV.

Let us mention in passing that in ultrarelativistic heavy ion collisions also quite spectacular explosions are observed, with radial and elliptic flows surprisingly well described by ideal hydrodynamics. The matter (sQGP) also seems to have [12] a very small viscosity, *i.e.* $\eta/\hbar s \approx$ 0.1-0.2 [13] and 0.2-0.4 from lattice simulations [14]. It is not even far from the lower CFT limit (2). Furthermore two of us suggested [1] that the sQGP has such low viscosity because of the existence of weakly bound states near the so called "zero binding lines" on the QCD phase diagram[†].

III. BOUNDS ON THE TRANSPORT PROPERTIES

In weak coupling the viscosity and diffusion coefficients are both related to the scattering length and are thought of as proportional to each other. For liquids one should think differently. An example of quite an opposite relation between them was provided by the famous Einstein relation, which we now derive for consistency in the presentation.

The distribution of suspended particles of mass m in a thermalized column of gas is given by statistical mechanics. Indeed, if $n(z)$ is the suspension density at finite temperature then

$$\frac{n(z)}{n(0)} = e^{-mgz/k_BT} \tag{7}$$

which follows from Boltzmann distribution. Einstein observed that an arbitrary sphere of radius r_0 in suspension within the column would also follows the same "distribution" profile. The idea is that the sphere under gravity will fall with a terminal Stokes velocity

$$v_t = \frac{mg}{6\pi\, r_0\, \eta} \tag{8}$$

but the fall will be balanced by random upward kicks due to Brownian motion. In equilibrium, the upward diffusion balances the downward gravitational fall so that in the stationary limit

$$D\,\frac{dn}{dz} = -\,n\, v_t\,, \tag{9}$$

where D is the diffusion constant. It from Eq. (9) that $n(z)/n(0) = e^{-v_t\, z/D}$. Comparing this result with (7) and using (8) yields the Einstein's formula

$$D = \frac{k_B\, T}{6\pi\, r_0\, \eta}\,, \tag{10}$$

in which D and η are related inversely. Although this formula was derived for a macroscopic sphere of radius r_0 immersed in a suspension, empirically it is known to hold through fourteen orders of magnitude changes down to the suspension constituent wavelength [15].

We recall that in three dimensions the diffusion constant is just $D = v^2\tau/3 = l_{mfp}^2/(3\tau)$, where l_{mfp} and τ are the mean-free path and collision time [‡]. Inserting this result into (10) yields

$$\frac{\eta}{1/(r_0 l_{mfp}^2)} = \frac{k_B\, T\tau}{2\pi}\,. \tag{11}$$

In a densely packed liquid the smallest jump (the mean-free path l_{mfp}) is the size of the quasiparticles r_0. Classically in densely packed hard balls τ can be as small as zero due to the fact that they are always touching. Quantum mechanically however this is not allowed since the time localization cannot be better than the limit set

[†] It was gratifying to learn subsequently that similar role for trapped atoms was played by Feshbach resonances.

[‡] We note that in d space-dimensions the diffusion constant is $D = l_{mfp}^2/\tau d$ and (10) should be derived accordingly. All the bounds to follow can be extended readily to d space dimensions.

by the *largest allowed energy*, by the Heisenberg uncertainty principle, i.e. $k_B T\,\tau \geq \hbar/2$. Inserting this result into (11) yields (2) since the entropy per unit volume s is just the number of (quasi)particles per unit volume due to the close packing, i.e. $s/k_B = n = 1/l_{mfp}^3$. The ensuing physical picture of the strongly coupled thermal system is that of a liquid with a shortest time correlation length $\tau_{min} = \hbar/2k_B T$.

In summary, our heuristic derivation of (2) follow from the assumption that (10) holds for the liquid particles, since the relation is known to hold over many orders of magnitude changes in η, D. Thus the particle and entropy densities are the same. While classically the collision time is zero for the densely packed liquid, quantum mechanically it is bounded from below by the Heisenberg uncertainty principle. Thus,

$$\frac{\eta}{\hbar\, s} \geq \frac{1}{4\pi\, k_B} \tag{12}$$

which is the same as the CFT limiting value. Turning the argument around through (10) implies an upper bound on the diffusion constant in a strongly coupled liquid, namely

$$\frac{D}{\sigma} \leq \frac{k_B T}{6\,\hbar} \tag{13}$$

in three dimensions, where the cross section $\sigma = 4\pi\, r_0^2$.

Let us now turn to cold atomic gases and repeat the same argument once more. It is simpler to imagine a Fermi gas in a vertical gravity field, for which we will rerun Einstein's derivation. We note that the trap field actually fulfills the same role. Using either Thomas-Fermi approximation or hydrostatic calculations we find that the Fermi momentum for a non-relativistic quasiparticle in an arbitrarily weak gravitational field is

$$p_F = \sqrt{2m^*(\mu - mgz)}\,, \tag{14}$$

where m^* is the fermions quasiparticle mass, m is their bare gravitational mass, and μ is the chemical potential. The normal Fermi density in an arbitrarily weak gravitational field is

$$\frac{n(z)}{n(0)} = \left(1 - \frac{mgz}{\mu}\right)^{3/2} \approx e^{-3mgz/(2\mu)} \tag{15}$$

A rerun of Einstein's preceding finite temperature arguments for the finite density column yields

$$\frac{\eta}{1/(r_0 l_{mfp}^2)} = \frac{1}{3\pi}\,(\mu\tau)\;, \tag{16}$$

with a diffusion constant

$$D = \frac{\mu}{9\pi\, r_0 \eta}\;. \tag{17}$$

In the infinite coupling limit l_{mfp} becomes r_0 and the system is again closely packed. The Heisenberg uncertainty principle stipulates that the shortest collision time is dictated by the largest available quasiparticle energy, namely $\mu\,\tau \geq \hbar/2$. Thus the new bound on the viscosity

$$\frac{\eta}{\hbar\, n} = \alpha_\eta \geq \frac{1}{6\pi}\;. \tag{18}$$

Although (18) was derived for non-relativistic particles, its insensitivity to the quasiparticle velocities imply that it should hold in the relativistic case as well. Also (18) implies an upper bound on the diffusion constant

$$\frac{D}{\sigma} \leq \frac{\mu}{18\,\hbar}\;, \tag{19}$$

in trapped cold Fermions in three dimensions.

IV. THE VISCOSITY OF STRONGLY COUPLED FERMIONIC ATOMS

Recent experiments of Kinast *et al.* [10] and Bartenstein *et al.* [11] provide direct access to transport properties of a cold Fermi gas of ^6Li atoms in the vicinity of the Feshbach resonance, *i.e.* in the strongly interacting regime. These experiments measure the frequencies and damping rates of the small collective oscillations of the atoms trapped in an external potential well.

Before we continue, we should mention that these are pioneering experiments and it is difficult to tell whether some differences between their results are due to somewhat different parameters of their systems or other experimental effects. Moreover, due to the anisotropic (*cigar-shaped*) form of the trapping potential the two collective modes in question–the axial and the radial—have very different frequencies. While the *softer* axial mode seems to be well described by hydrodynamics, the radial one which has an order of magnitude higher frequency exhibits deviations from hydrodynamics and is not yet fully understood.

If the collision rate of atoms is large enough to establish local equilibrium the collective vibrations of the atomic cloud can be described using standard hydrodynamical theory [16]. The collective vibrations are described by the local density $n(\vec{r})$, pressure $p(\vec{r})$ and velocity $\vec{v}(\vec{r})$ which are the solutions of the continuity equation, Euler equations of motions and the equation of state:

$$\begin{aligned} &m\,\frac{\partial n}{\partial t} + \nabla\cdot(m\,n\,\vec{v}) = 0\,, \\ &m\,n\,\frac{\partial \vec{v}}{\partial t} + m\,n\;(\vec{v}\cdot\nabla)\;\vec{v} = -\nabla p - n\,\nabla V\,, \\ &p = A\,n^{\gamma+1}\,, \end{aligned} \tag{20}$$

where $V = (1/2)\,m\,\sum_i \omega_i^2\, r_i^2$ is the harmonic potential in which atoms are trapped, A is a constant, and γ is the polytropic index.

Although one of our goals is to describe viscous damping of the collective oscillations via Navier-Stokes equations, we will do it perturbatively, starting from known solution of Euler equations.

The lowest collective modes correspond to the small vibrations of the density, pressure and velocity around their equilibrium values, n_{eq}, p_{eq} and $\vec{v}_{eq}=0$. These values are determined by the static limit of the Eqs. (20) in which the Euler equation takes the form $\nabla p_{eq} + n_{eq}\nabla V = 0$. The equilibrium density is

$$n_{eq}(r) = n_{eq}(0)\left(1 - \sum_{i=1}^{3}\frac{r_i^2}{R_i^2}\right)^{1/\gamma}, \qquad (21)$$

for $\vec{r}$ inside the ellipsoid

$$\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} + \frac{z^2}{R_z^2} = 1 \qquad (22)$$

and zero outside. In equation (21) $n_{eq}(0)$ is the equilibrium density of the atomic cloud at the center of the harmonic trap ($\vec{r}=0$) and

$$R_i = \sqrt{\frac{p_{eq}(0)}{n_{eq}(0)}\,\frac{2(\gamma+1)}{\gamma\, m\omega_i^2}} \qquad (23)$$

are the radii of the cloud with $p_{eq}(0)$ being the equilibrium central value of the pressure. These radii can be expressed in terms of the global chemical potential of the cloud μ, which for N spin-1/2 fermions in a harmonic trap is defined as [16]

$$\mu = \hbar\bar{\omega}\,(3N)^{1/3} = k_B\, T_F\,, \qquad (24)$$

where $\bar{\omega}^3 = \omega_1\,\omega_2\,\omega_3$ and k_B is the Boltzmann constant. The above equation also defines the Fermi temperature T_F. Using the Gibbs-Duham relation, $dp = n\, d\mu$ (valid at constant temperature) and the equation of state, $p = An^{\gamma+1}$, one can easily show that

$$\frac{p}{n} = \frac{\gamma}{\gamma+1}\,\mu$$

so that the Thomas-Fermi radii are

$$R_i = \sqrt{\frac{2\,\mu}{m\omega_i^2}}\,, \qquad (25)$$

independent of γ.

The central equilibrium density $n_{eq}(0)$ in Eq. (21) can be found by hydrostatics or using Thomas-Fermi approximation with a local Fermi energy $\epsilon_F(r) = \hbar^2 k_F^2(r)/2m$, where $\hbar\, k_F(r)$—a local Fermi momentum—is related to the chemical potential and the trap energy by

$$\epsilon_F(r) + V(r) = \mu\,. \qquad (26)$$

Since the density of the Fermi liquid is $n{=}\hbar^3 k_F^3/3\pi^2$ the central density is

$$n_{eq}(0) = \frac{1}{3\pi^2}\left(\frac{2m\,\mu}{\hbar^2}\right)^{3/2}. \qquad (27)$$

Linearizing Eqs. (20) to describe small density $n = n_{eq} + \delta n\, e^{i\omega t}$ and velocity $\vec{v}\, e^{i\omega t}$ oscillations one can obtain the following equations for the density and velocity amplitudes:

$$\begin{aligned} -m\omega^2\delta n &= \nabla\cdot\left(n_{eq}\nabla\left(\frac{1}{n_{eq}}\frac{dp}{dn}\delta n\right)\right) \\ -\omega^2\vec{v} &= \nabla\left(\vec{v}\cdot\nabla V\right) + \gamma\,\nabla\cdot\vec{v}\,\nabla V\,. \end{aligned} \qquad (28)$$

The form of the collective (hydrodynamic) vibrational modes which are the solutions of Eqs. (28) depend on the symmetry of the trap potential. If the confining potential is isotropic, $\omega_1=\omega_2=\omega_3$, the collective modes have spherical symmetry and can be characterized by definite angular momentum and its z component, l and m. The monopole mode, $l=m=0$, has a velocity profile which is proportional to $\vec{r}$. Such modes are referred to as breathing modes. The dipole mode $l=1$ involves the motion of the center of mass of the cloud and is not usually excited in the experiments we are interested in. There are five degenerate quadrupole modes corresponding to $l=2$ with $m=0,\pm1,\pm2$.

Here we will discuss the collective modes that are excited only in a axially symmetric trap with $\omega_1=\omega_2=\omega_r$ and $\omega_3=\omega_z=\lambda\omega_r$, where λ is a constant. In the experiment of Bartenstein *et al.* [11] λ is 0.03[§]. In such traps the collective modes corresponding to different angular momenta but the same z-component m are mixed.

To find these modes we look for the solutions of Eqs. (28) in the form[**]

$$\vec{v} = (a_x x, a_y y, a_z z) = (a_r x, a_r y, a_z z)\,. \qquad (29)$$

The set of equations (28) for $\vec{v}$ reduces to the secular equation for the eigenfrequencies and the corresponding eigenvectors. The three frequencies are

$$\begin{aligned} \omega_{1,2}^2 &= \omega_r^2(1+\gamma+\frac{1}{2}(\gamma+1)\lambda^2 \\ &\quad \pm\frac{1}{2}\sqrt{(\gamma+2)^2\lambda^4+(\gamma^2-3\gamma-2)\lambda^2+4(\gamma+1)^2})\,, \\ \omega_3 &= \sqrt{2}\,\omega_r\,, \end{aligned} \qquad (30)$$

where in the first equation the plus and minus signs correspond to axial and radial modes with frequencies $\Omega_z=\omega_1$ and $\Omega_r=\omega_2$ respectively. The third frequency (which is the same as one of the frequencies in the case of the spherical trap) is the frequency of the two remaining degenerate modes with $l=2$, $m=\pm1\pm2$. For the cigar-shaped traps, *i.e.* in the limit of a very small λ, the axial and radial frequencies reduce to

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[§]Such traps with $\lambda \ll 1$ are referred to as cigar-shaped or prolate.

[**]Such a flow with velocity components proportional to the position is often referred to as Hubble flow.

$$\Omega_z = \omega_z \sqrt{3 - \frac{1}{\gamma+1}}\,, \quad \Omega_r = \omega_r \sqrt{2\gamma+2}\,. \tag{31}$$

As discussed in [8] a cold gas of strongly interacting fermions at the point of the Feshbach resonance has a universal equation of state as in the third equation in Eqs. (20) with polytropic index $\gamma = 2/3$. With this value of γ the axial and radial frequencies are

$$\frac{\Omega_z}{\omega_z} = \sqrt{\frac{12}{5}} = 1.55\,, \quad \frac{\Omega_r}{\omega_r} = \sqrt{\frac{10}{3}} = 1.83\,. \tag{32}$$

The corresponding velocities in the same limit ($\lambda \to 0$) are given for the axial and radial modes, respectively, by

$$\begin{aligned} \frac{a_z}{a_r} &= -\frac{2+2\gamma}{\gamma\,\lambda^2} = -5\left(\frac{\omega_r}{\omega_z}\right)^2, \\ \frac{a_z}{a_r} &= \frac{1}{1+\gamma} = \frac{2}{5}\,, \end{aligned} \tag{33}$$

where $\gamma = 2/3$ was used.

The above frequencies can be compared to the experimental values. The value of $\gamma = 2/3$ gives the axial frequency

$$\frac{\Omega_z}{\omega_z} = \sqrt{\frac{12}{5}} = 1.55\,. \tag{34}$$

which perfectly agrees with data [11] at the point of the Feshbach resonance. However, the corresponding prediction for the frequency of the radial mode,

$$\frac{\Omega_r}{\omega_r} = \sqrt{\frac{10}{3}} = 1.83 \tag{35}$$

is about 20% larger then the observed value of 1.67 [11]. The failure of hydrodynamics to describe the observed value of the radial mode can be attributed to the fact that the large frequency of the mode prevents the onset of the local equilibrium required for the hydrodynamics to applicable. Estimates of the collision time τ in section V show that indeed $\Omega_r\tau \approx 1$ so that hydrodynamics is not applicable. This conclusion is also supported by sudden jumps in the frequency and especially the dumping rate of the radial mode [11] not far from the resonance on the "BCS" side. There is no visible jumps at this particular value of the scattering length in the axial mode, so this phenomenon cannot possibly be associated with a change in the equation of state or transport properties by itself. An extra source of dissipation in the radial mode must thus be associated with a non hydrodynamical effects[††].

Let us now apply viscous hydrodynamics with the universal equation of state to damping of the modes, by considering dissipation effects in the vicinity of the Feshbach resonance. The primary source of dissipation in the hydrodynamic limit is shear viscous flow [19]. The rate of change of the energy of a mode is given by [20]

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle &= -\int \frac{\eta}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\,\delta_{ik}\,\nabla\cdot\vec{v}\right)^2 d^3r \\ &= -\frac{2}{3}\,a_r^2\left(1 - \frac{a_z}{a_r}\right)^2 \int \eta(r)\,d^3r\,, \end{aligned} \tag{36}$$

where η is the coefficient of the shear viscosity and the ratio a_z/a_r is given in Eqs. (33) for each mode. As was mentioned above while the viscous terms in the equations of motion Eqs. (20) vanish for the Hubble-like flow, Eq. (29), the rate of change of energy depends on the first derivatives of the velocity. Thus, while the viscous forces between different fluid elements vanish for the modes with Hubble-like velocity the energy still dissipates at the rate given by Eq. (36).

A damping rate of a collective mode can now be obtained by dividing the rate of change of energy, Eq. (36) by half of the the time-averaged total energy of the vibrational mode which is equal to maximum kinetic energy of the mode

$$\begin{aligned} \langle E \rangle &= \frac{m}{2}\int n_{eq}(\vec{r})\,v^2(\vec{r})\,d^3r \\ &= \frac{\pi^2}{128}\,m\,n_{eq}(0)\,a_r^2\,R_x\,R_y\,R_z\left(R_x^2 + R_y^2 + \frac{a_z^2}{a_r^2}\,R_z^2\right) \\ &= \frac{1}{16}\,m\,N a_r^2\left(2\,R_r^2 + \frac{a_z^2}{a_r^2}\,R_z^2\right), \end{aligned} \tag{37}$$

where Eq. (21) was used with $\gamma = 2/3$. In the last step the product of Thomas-Fermi radii was expressed in terms of the total number of particles N in the cloud,

$$\begin{aligned} N &= \int n_{eq}(\vec{r})\,d^3r = \int n_{eq}(0)\left(1 - \sum_{i=1}^{3}\frac{r_i^2}{R_i^2}\right)^{3/2} d^3r \\ &= \frac{\pi^2}{8}\,n_{eq}(0)\,R_x\,R_y\,R_z\,, \end{aligned} \tag{38}$$

and $R_x = R_y = R_r$ for the axially symmetric trap.

Using Eqs. (36) and (37) the damping rate is

$$\begin{aligned} \Gamma &= \frac{1}{2}\left|\frac{\langle dE/dt\rangle}{\langle E\rangle}\right| \\ &= \frac{16}{3\,m\,N}\,\frac{\left(1 - \frac{a_z}{a_r}\right)^2}{\left(2\,R_r^2 + \frac{a_z^2}{a_r^2}\,R_z^2\right)}\int \eta(r)\,d^3r\,. \end{aligned} \tag{39}$$

Thus, the damping rate is proportional to the volume-integrated viscosity. Ignoring the temperature, $T = 0.03T_F$ in the experiment of Bartenstein *et al.* [11], in the spirit of universality we use the relation (6), reducing

[††]We are grateful to R. Grimm for email in which he informed us about a possible source of this effect.

the integral to the total number of particles times the coefficient α_η we want to determine, so that finally [‡‡],

$$\Gamma = \frac{16}{3\,m}\,\frac{\left(1-\frac{a_z}{a_r}\right)^2\,\hbar\,\alpha_\eta}{\left(2\,R_r^2+\frac{a_z^2}{a_r^2}\,R_z^2\right)}\,. \quad (40)$$

For the axial mode in the cigar-shaped potential trap the ratio a_z/a_r is much larger than one (Eq. (33)), so that the coefficient α_η to a good approximation is

$$\alpha_\eta \approx \frac{3\,m\,R_z^2\,\Gamma_z}{16\hbar} \quad (41)$$

The experimental values are $\Gamma_z/\omega_z = 0.0036$ at the point of the Feshbach resonance, with the minimal value of $(\Gamma_z/\omega_z)|_{min} = 0.0015$ slightly off the resonance. Using the absolute axial trap frequency $\omega_z \approx 140\,Hz$ we finally get our result for the minimal dimensionless viscosity[§§]

$$\alpha_\eta^{(z)}|_{min} \approx 0.5\,. \quad (42)$$

It is also instructive to extract the value of α_η from the radial mode. Using Eq. (33) and the fact that $R_z = R_r/\lambda$ we get from Eq. (40),

$$\alpha_\eta \approx \frac{m\,R_x^2\,\Gamma_r}{12\hbar\,\lambda^2}\,. \quad (43)$$

In the experiment of Bartenstein *et al.* [11] with the radial trap frequency $\omega_r \approx 4700\,Hz$ the damping rate of the radial mode at the Feshbach resonance is $\Gamma_r = 0.0625\,\omega_r$. Thus, the coefficient α_η extracted from the damping rate of the radial mode is much larger

$$\alpha_\eta^{(r)} \approx 60\,, \quad (44)$$

which two orders of magnitude above the minimal value extracted from the damping of the axial mode. As discussed above (as well as in the next section), it only proves that the radial mode is not a hydrodynamical mode so one should not be surprised that its damping is not described by viscous hydrodynamics.

V. COMPARISON WITH THE TRADITIONAL KINETIC THEORY

In this section we will use the usual kinetic theory, based on the notion of binary collisions, and show that it completely fails to describe transport properties of the system. This should be a convincing evidence that the cloud of cold atoms in the experiment of Bartenstein *et al.* [11] is a near perfect quantum liquid, with a dissipation which is nearly the smallest allowed in nature, rather then a dilute gas of atoms.

We start with an order of magnitude estimates of the collision rates and viscosity using the most naive approach without Pauli blocking and with the largest "unitary limited" cross section $\sigma = 4\pi/k_F^2$. The collision rate at the center of the trap estimated like this gives

$$\tau_{coll}^{-1} = n(0)\sigma v_f \sim 10^5 s^{-1} \quad (45)$$

where the last number corresponds to the conditions of the experiment of Bartenstein *et al.* [11]. Comparing with oscillation frequencies, $\omega_r = 4712\,Hz$ and $\omega_z = 142\,Hz$ of the trap, leads to a conclusion that only the latter mode has a chance to be hydrodynamical.

The mean free path $l_{\rm mfp}$, of a particle is of order $(n\sigma)^{-1}$, while the shear viscosity is

$$\eta \sim m\,\bar v\,n\,l_{\rm mfp} = \frac{m\,v}{\sigma}\,, \quad (46)$$

where v is the average velocity of a particle. In the limit of zero temperature the velocity is set by the Fermi momentum, $m\,v_F = \hbar\,k_F$. In the vicinity of the Feshbach resonance the cross section is unitary bounded, $\sigma < \sigma_{max} = 4\pi/k_F^2$. So if we take its *maximal* value (and still ignore Pauli blocking), we get a *minimal* viscosity which may follow from binary collisions

$$\frac{\eta}{\hbar\,n} > \frac{40}{6\pi}\,. \quad (47)$$

This inequality is strongly violated in experiment, as shown above: this "minimal binary" value is in fact four times the observed one, and forty times larger than the bound for a liquid.

Furthermore, since we speak about fermionic atoms, the collision rate will be significantly lowered by Pauli blocking which should lead to a suppression factor of about $(T/T_F)^2 \sim 1/1000$ in the experimental conditions. If true, the oscillations then would be basically collisionless and no hydro-phenomena would be present. In a picture of BCS-type pairing, with relatively small modification of the Fermi sphere, T in the above formula is substituted by a gap, so the rescattering suppression would be of the order of $(\Delta/\epsilon_F)^2$. The gap value is not well known, but this suppression factor is still about 1/100 or so. We must then conclude that both pictures are

[‡‡] The universal relation should not be valid for too dilute parts near the edges of the system, at less than one mean-free-path or at optical depth less than 1, where dissipation is larger. However that edge includes only about 1 percent of particles under the experimental conditions, and is thus ignored.

[§§] The error is comparable to the value itself, as can be seen from experimental data points. Ironically, the situation with the dimensionless viscosity of quark-gluon plasma is quite similar.

wrong and in fact there seems to be no Pauli blocking whatsoever***.

To make a more quantitative conclusion we will derive here the damping rate of a collective mode in an axially symmetric trap applying the traditional kinetic equation to an almost degenerate Fermi gas with unitary limited cross section.

A damping rate of a collective mode in the kinetic theory is determined by a relaxation time which is a measure of how fast a particle distribution function $n(\vec{p},\vec{r},t)$ for a given collective mode takes an equilibrium form. Both the time dependent and equilibrium distribution functions are the solutions of the kinetic equation. The equilibrium distribution for a Fermi gas is

$$n(\epsilon,\vec{r}) = \left(e^{(\epsilon-\mu+V(r))/k_B\,T}+1\right)^{-1}\,, \tag{48}$$

where $\epsilon = p^2/2m$.

During an oscillation the distribution function is different from the equilibrium one. The collisions between particles cause the non-equilibrium distribution function to "relax" to the equilibrium form. These collisions are the source of the damping of the oscillations.

As shown in [16] the damping rate of the oscillations of Fermi gases is equal to

$$\Gamma = \frac{\left\langle \left(p_{1,z}^2 - p_1^2/3\right)\Gamma\left[p_{1,z}^2-p_1^2/3\right]\right\rangle}{\left\langle p_{1,z}^2 - p_1^2/3\right\rangle}\,, \tag{49}$$

where

$$\begin{aligned}
&\left\langle \left(p_{1,z}^2 - p_1^2/3\right)\Gamma\left[p_{1,z}^2-p_1^2/3\right]\right\rangle \\
&= \frac{1}{4(2\pi\hbar)^6}\int d^3r\,d^3p_1\,d^3p_2\,d^3p_1'\,d^3p_2'\,(\Delta\Phi)^2 \\
&\quad \delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_1{}'-\vec{p}_2{}')\,\delta(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2') \\
&\quad W(\vec{p}_1,\vec{p}_2;\vec{p}_1{}',\vec{p}_2{}')\,n_1\,n_2\,(1-n_1')\,(1-n_2')\,,
\end{aligned} \tag{50}$$

and,

$$\left\langle \left(p_z^2-p^2/3\right)^2\right\rangle = \int d^3r\,d^3p\,(\Phi)^2\,n\,(1-n) \tag{51}$$

where $\Delta\Phi = (\Phi_1+\Phi_2)-(\Phi_2'+\Phi_2')$ and the function $\Phi = p_z^2 - p^2/3$ describes the deviation of the distribution function of a collective mode from the equilibrium one. The function $W(\vec{p}_1,\vec{p}_2;\vec{p}_1{}',\vec{p}_2{}')$ is given by the scattering amplitude of binary collisions $(\vec{p}_1\,,\vec{p}_2 \to \vec{p}_1{}'\,,\vec{p}_2{}')$. In the vicinity of the Feshbach resonance this function is determined by the unitary limit of the scattering amplitude and is equal to

$$W = \frac{\hbar^2}{m^2}\,\frac{(2\pi\hbar)^3}{p^2} \tag{52}$$

$\vec{p} = \vec{p}_1 - \vec{p}_2$ is the relative momentum of two particles.

At low temperatures the Pauli blocking factors, $n\,(1-n)$, in Eqs. (50) and (51) significantly reduce the phase space of particles whose collisions appreciably contribute to the relaxation. The main contribution is from the collisions of particles whose momenta lie very close to the Fermi surface:

$$\epsilon - \mu + V(r) \sim k_B\,T\,, \tag{53}$$

where μ is the chemical potential Eq. (24).

After a lengthy but straightforward calculation one obtains for the damping rate

$$\Gamma \approx \frac{9\pi}{50}\,\frac{(k_B\,T)^2}{\hbar\,\mu}\,. \tag{54}$$

Note the temperature dependence of the damping rate. It has a typical T^2 dependence which comes from the life time of weakly interacting Fermi particles or quasi-particles in the case of the Fermi liquid [20]. We stress that such scaling is true only for weakly interacting gas of particles or quasi particles which is observed in liquid ^3He. If the atomic cloud near the Feshbach resonance is indeed a strongly interacting near perfect liquid the damping rate and other dissipative processes will very weakly depend on the temperature. This prediction can be checked experimentally.

The damping rate can be expressed in terms of the trap frequencies and the Fermi temperature T_F as

$$\Gamma \approx \frac{9\pi}{50}\,(3N)^{1/3}\,\bar{\omega}\left(\frac{T}{T_F}\right)^2\,. \tag{55}$$

In the experiment of Bartenstein *et al.* [11] the atomic cloud has $N = 4\times10^5$ particles, a temperature $T \approx 0.03T_F$, with $T_F = 1.2\times10^{-6}\,K$, and an average trap frequency $\bar{\omega} \approx 1470\,Hz$. With these values the above formula for the axial mode gives

$$\frac{\Gamma_z}{\omega_z} \approx 0.56\,, \tag{56}$$

This is much larger than the damping observed close to the Feshbach resonance, and is only compatible with data well away from it, where $k_F\,|a| \ll 1$. Thus, we conclude that a strongly interacting atomic cloud near the Feshbach resonance is described better by a nearly perfect strongly interacting limit with viscosity very close to the minimum possible value. As one moves away from the Feshbach resonance into the regime of a weakly interacting Fermi gas, kinetic theory becomes again applicable. Furthermore, as detuning gets larger than used in the experiment, the scattering length becomes smaller, and the gas enters an almost collisionless regime with a small damping rate. Thus, the damping rate is expected to

***After observing the elliptic flow this issue was discussed in the literature and the MIT group [17] has argued that this may be due to a strong deformation of the Fermi sphere in the exploding gas. This explanation obviously is not applicable to our small amplitude oscillation.

reach a maximum value for a certain value of the magnetic field.

For the radial mode the value predicted by the same kinetic calculation is

$$\frac{\Gamma_z}{\omega_z} \approx 0.02\,, \tag{57}$$

which is comparable to the value of about 0.06 obtained experimentally. It shows again that this mode is not supposed to be treated by hydrodynamics, while its traditional kinetic treatment is reasonable.

VI. CONCLUSIONS AND DISCUSSION

Strongly interacting systems, both in field theories (QCD,CFL) and condensed matter physics (strongly coupled plasmas, Feshbach atoms), are radically different from gas-like weakly interacting systems. The best way to see that is *not* via the equation of state and related bulk thermodynamical observables, but with the help of transport properties. Transition from collisionless to collisional regime observed (and emphasized in this work) is completely incompatible with Fermi gas/liquid ideology, as it predicts strong Pauli blocking.

As we have argued above, all such systems are near-perfect liquids, and thus the natural tool one should use to describe them is standard viscous hydrodynamics. Indeed, one gets very good description of the "elliptic flow" in a released trap, or the frequency of the softest oscillations.

Furthermore, the viscosity extracted from the data are shown to be very different from what is expected on the basis of binary collisions and weak coupling regimes. The minimal value of the viscosity (or maximal rescattering rate) (42) observed experimentally was compared to results of standard kinetic theory, and even without Pauli blocking and with maximal (unitary limited) binary cross section it fails to get even close to data. We have shown that standard kinetic theory provides reasonable damping away from the Feshbach resonance, where the return to the usual gas-like regime takes place. A system of trapped cold atoms, like other strongly coupled systems mentioned in the introduction, is not a gas but rather a near-perfect liquid.

What can be the *most* perfect liquid? At infinite coupling the constituents are effectively large and densely packed. The packing fluctuates over short time scales dictated by the Heisenberg uncertainty principle. These physical insights together with Einstein's description of diffusion in viscous liquids, allows a simple re-derivation of the viscosity to entropy density ratio established using CFT. We have extended this derivation to cold Fermi systems and derived an even lower bound for the viscosity to the particle density ratio.

We now proceed to discuss what further experimental and theoretical work should be done to clarify the present issues. We think it would be quite important to make more accurate damping measurements in order to see how close is the minimal viscosity available experimentally to the theoretical bound. The issue of Pauli blocking would certainly be better understood if such damping would be studied for a range of temperatures. We expect that the temperature dependence is weak and is not even close to what the binary collision theory for Fermi gas predicts. The system is not even qualitatively close to a Fermi gas at strong coupling.

The temperature dependence is also crucial for understanding the issue of superfluidity, which was carefully avoided in this paper. It seems that the lowest oscillation mode is well described by the usual one-liquid hydrodynamics. However, it is of course quite likely that some higher excitations can be analogous not to the usual sound but to other sounds known for superfluid liquid ^4He.

Theorists should of course try to develop a theory of the strongly coupled systems beyond quite schematic mixtures of the BCS superconductor and ideal Bose gas of molecules. So far, such models can approximately reproduce the equation of state but not the transport properties. Numerical simulations of larger scales can also be helpful. Perhaps one should complement the equation of state calculations made so far by measurements of long-time correlators related by Kubo relations to transport coefficients. This is quite insightful in the description of classical strongly coupled plasma.

Acknowledgments.

We thank Gerry Brown for numerous and very helpful discussions on Fermi liquids. We also thank H.T.C. Stoof for a useful discussion at the early stage of this work and R. Grimm and J.E. Thomas for helpful correspondence. This work was partially supported by the US-DOE grants DE-FG02-88ER40388 and DE-FG03-97ER4014.

[1] E.V. Shuryak and I. Zahed, Phys. Rev. **C70** (2004) 021902, `hep-ph/0307267`; for a recent review see E.V. Shuryak, `hep-ph/0405066`.

[2] S. Ichimaru *et al.*, Phys. Rep. **149C** (1987) 91.

[3] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231, `hep-th/9711200`; Phys. Rev. **80** (1998) 4859, `hep-th/9803002` and references therein.

[4] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, Nucl. Phys. **B534** (1998) 202; C.P. Burgess, N.R. Constable, R.C. Myers, JHEP (1999) 9908; C. Kim and S.J. Rey, Nucl. Phys. **B564** (2000) 430.

[5] G. Policastro, D.T. Son and A.O. Starinets, Phys. Rev. Lett. **87** (2001) 081601.

[6] A. Buchel, J.T. Liu and A.O. Starinets, hep-th/0406264.
[7] A.O. Starinets, Phys. Rev. **D66** (2002) 124013, hep-th/0207133.
[8] H. Heiselberg, Phys. Rev. **A63** (2001) 043606.
[9] K.M. O'Hara *et al.*, Science **298** (2002) 2179; T. Bourdel *et al.*, Phys. Rev. Lett. **91** (2003) 020402.
[10] J. Kinast *et al.*, Phys. Rev. Lett. **92** (2004) 150402.
[11] M. Bartenstein *et al.*, Phys. Rev. Lett. **92** (2004) 203201.
[12] D. Teaney, J. Lauret and E.V. Shuryak, Phys. Rev. Lett. **86** (2001) 4783, nucl-th/0011058; for details see nucl-th/0110037.
[13] D. Teaney, J. Phys. **G30** (2004) S1247, nucl-th/0403053.
[14] A. Nakamura and S. Sakai, hep-lat/0406009.
[15] X. Xia and P.G. Wolynes, Phys. Rev. Lett. **86** (2001) 5526, cond-mat/0101053.
[16] C.J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, Cambridge University Press, 2002.
[17] S. Gupta *et al.*, Phys. Rev. Lett. **92** (2004) 100401.
[18] H. Heiselberg, Phys. Rev. Lett. **93**, (2004) 040402, cond-mat/0403041.
[19] G.M. Kavoulakis, C.J. Pethick and H. Smith, Phys. Rev. **A57** (1998) 2938.
[20] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 1959.